SPECTRAL ANALYSIS

Due date: 04/02/2020

Fourier transforms are one of the most commonly used signal processing tools. In a nutshell, think of a time-domain function $h(t)$ and a corresponding frequency-domain function $H(f)$ as two representations of a single function, with the following mapping that lets you go back and forth between the two:

$$H(f) = \int_{-\infty}^{\infty} h(t) e^{i2\pi ft} dt, \quad h(t) = \int_{-\infty}^{\infty} H(f) e^{-i2\pi ft} df.$$

For N discrete data points $h(t_k) \equiv h_k$, the discrete Fourier transform is given by:

$$H(f_n) \equiv H_n = \sum_{k=0}^{N-1} h_k e^{i2\pi kn/N},$$

and its corresponding inverse is given by:

$$h(t_k) \equiv h_k = \frac{1}{N} \sum_{n=0}^{N-1} H_n e^{-i2\pi kn/N}.$$

Because of the Fast Fourier Transform (FFT), a range of operations that would otherwise be prohibitively slow can now be done. Two most notable examples are convolution:

$$r(t) * s(t) = \int_{-\infty}^{\infty} r(t) s(\tau - t) dt,$$

and correlation:

$$r(t) \star s(t) = \int_{-\infty}^{\infty} r(t) s(t + \tau) dt,$$

where τ is the lag parameter. The discrete versions of these two equations are:

$$(r * s)_j = \sum_{k=-N/2+1}^{N/2} s_{j-k} r_k, \quad (r \star s)_j = \sum_{k=0}^{N-1} r_{j+k} s_k.$$

If the signal function s_j is periodic with period N and the response function r_k is *finite* on the $[-N/2, N/2]$ interval, then their convolution and correlation can be computed using FFT:

$$\mathcal{F}(r * s) = \mathcal{F}(r)\mathcal{F}(s), \qquad \mathcal{F}(r \star s) = \mathcal{F}(r)\mathcal{F}^*(s).$$

a) Compute and plot the Fourier transform and the inverse Fourier transform for the following functions: sine, cosine, square pulse, triangle, delta. Make sure the axes are properly labeled and make sense. Compute the (one-sided) power spectral density for all functions. Again watch for the axes. This should give you a good idea of what the Fourier transform does.

b) Create a frequency-rich signal of your choice and demonstrate the effects of aliasing.

c) Analyze the sound of a guitar (the wav and the text files are on the course homepage).

d) Listen to Bach's Partita in the uncompressed mp3 format. Then listen to a 2.3 s exerpt that has been sampled at different rates: 882, 1378, 2756, 5512, 11025 and 44100 Hz. Once you hear what the effect of undersampling is, quantify it by the Fourier transform. All data (in both mp3 and ascii) are available on the course webpage.

e) A study of car density as function of time on the turnpike exit to the Dodgers stadium in LA was performed in a period of roughly 25 weeks, with a 5-min sampling. In addition, the start time, end time and the number of visitors was recorded for the events at the stadium. The data are in files `dodgers.cars.data` and `dodgers.events.data`. Analyze the data and interpret the results.

f) Compute the autocorrelation function of the sound of boiling water. If you are so inclined, acquire your own data, otherwise you can use `boiling.data` from the course homepage. *Hint:* the autocorrelation function may drop rapidly and you should use the log scale to visualize it.

g) Sunspots are closely correlated with the Sun's magnetic activity. Their number has been recorded since 1700 on a yearly basis and since 1749 on a monthly basis. Using FFT, find any periodicity in the data (`sunspots.yearly.data` and `sunspots.monthly.data`). Compute autocorrelation functions for both data-sets and compare them. Is higher sampling of sunspots warranted?

h) Using convolution, predict the shape of a spectral line from a diffraction-limited spectrograph. *Hint:* show (or assume) that the response function is a gaussian.